Stats100A **Summer 2024**

Week 5: Extra Problems - Solutions

Author: Andrew Lizarraga

Solutions

5.1 Expectation & Variance

Problem 1: I flip a fair coin 10 times. What is the expected number of heads?

Solution: Let X_i denote the ith flip, where $X_i = 1$ with probability $p = \frac{1}{2}$ and 0 with probability $1 - p = \frac{1}{2}$. Thus by linearity of expectation, we have:

$$E(X_1 + \cdots + X_{10}) = E(X_1) + \cdots + E(X_{10}) = \frac{10}{2} = 5$$

Problem 2: How many times would I expect to flip a fair coin until I see a heads?

Solution: Denote this random variable by X. Notice that X is a geometric random variable, thus the expectation is given by:

$$E(X) = \frac{1}{p} = \frac{1}{1/2} = 2$$

Remark: We can also deduce the expectation this way. Recall by the law of total expectation that we have:

$$E(X) = E(X|X = 1)P(X = 1) + E(X|X = 0)P(X = 0)$$
$$E(X) = \frac{1}{2} + (1 + E(X))\frac{1}{2}$$
$$2E(X) = 2 + E(X)$$
$$E(X) = 2$$

Problem 3: How many times would I expect to roll a die until I see a 5?

Solution: We solve this similarly to problem 2:

$$E(X) = \frac{1}{1/6} = 6$$

Problem 4: X is a discrete random variable with distribution $q(x)$ and assumes values from a up to $a+n$. What is it's expectation?

Solution: By definition:

$$E(X) = \sum_{x=a}^{a+n} xq(x)$$

Problem 5: X is a continuous random variable with distribution $q(x)$, with $q(x) > 0$ for $x \in [a,b]$, otherwise it's 0. What is the expectation of X ?

Solution: By definition:

$$E(X) = \int_a^b xq(x)dx$$

Problem 6: Given a random variable X, what is it's variance? Can you express the variance in two different ways?

Solution: By definition:

$$\text{Var}(X) = E[(X - E[X])^2]$$

Which can also be expressed as:

$$\begin{aligned} &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - 2E[X]^2 + E[X]^2 \\ &= E[X^2] - E[X]^2 \end{aligned}$$

Problem 7: Is it the case that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$?

Solution: In short No, unless X and Y are uncorrelated. To properly express the variance, consider the following computation:

$$\begin{aligned} \text{Var}(X+Y) &= E((X+Y)^2) - (E(X+Y))^2 \\ &= E(X^2 + 2XY + Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 \\ &= [E(X^2) - E(X)^2] + [E(Y^2) - E(Y)^2] - 2E(X)E(Y) + 2E(XY) \\ &= \text{Var}(X) + \text{Var}(Y) + 2[E(XY) - E(X)E(Y)] \end{aligned}$$

Problem 8: Does $E(XY) = E(X)E(Y)$?

Solution: No. Suppose $X = 1$ with probability $\frac{1}{2}$ and $X = -1$ with probability $\frac{1}{2}$. Also suppose that $Y = \frac{1}{X}$.
Then $E(XY) = 1$, but $E(X) = 0$ and $E(Y) = 0$, so $E(X)E(Y) = 0$, thus $E(XY) \neq E(X)E(Y)$.

Problem 9: I roll a fair 6-sided die once. Whatever value it lands on, call it a. Now roll a dice and take the sum of the face values rolled and call it b. What is $E(b)$?

Solution: Note that the expectation of a single fair die roll is $a = \frac{1+2+3+4+5+6}{6} = 3.5$. Now we roll 3.5 more dice on average and each of these dies have an expectation of 3.5 as well. Thus $b = (3.5)(3.5) = 12.25$.

Problem 10: Let X be a nonnegative integer-valued random variable and k a nonnegative constant. Show that $P(X \geq k) \leq \frac{E(X)}{k}$.

Solution: By definition of expectation we have:

$$E(X) = \sum_{x} xP(X = x)$$

And this can be expressed as:

$$E(X) = \sum_{x<k} xP(X = x) + \sum_{x \geq k} xP(X = x)$$

Notice that since x is nonnegative that:

$$E(X) \geq \sum_{x \geq k} xP(X = x)$$

$$E(X) \geq \sum_{x \geq k} xP(X = x) \geq \sum_{x \geq k} kP(X = x)$$

$$E(X) \geq k \sum_{x \geq k} P(X = x)$$

$$E(X) \geq kP(X \geq k)$$

$$\frac{E(X)}{k} \geq P(X \geq k)$$

Problem 11: Let X be a nonnegative random variable and k a nonnegative constant. Show that $P(|X - E(X)| \geq k) \leq \frac{\mathrm{Var}(X)}{k^2}$

Solution:
First let's make the following observations:

1. Notice that $P(|X - E(X)| \geq k) = P((X - E(X))^2 \geq k^2)$.
2. If we let $Z = (X - E(X))^2$, then Z is a nonnegative random variable.
3. From the law of total expectation we have:

$$E(Z) = E(Z|Z \geq k^2)P(Z \geq k^2) + E(Z|Z < k^2)P(Z < k^2)$$

Notice that $E(Z|Z \geq k^2) \geq k^2$ since we are conditioning on the fact that $Z \geq k^2$ so the expectation can't be less than k^2. Also since Z is nonnegative, we know that $E(Z|Z < k)P(Z < k)$ is also nonnegative and can be set to 0 in order to establish the inequality. So we have:

$$E(Z) \geq k^2 P(Z \geq k^2) + 0$$

$$\frac{E(Z)}{k^2} \geq P(Z \geq k^2)$$

Now substitute $Z = (X - E(X))^2$

$$\frac{E((X - E(X))^2)}{k^2} \geq P((X - E(X))^2 \geq k^2)$$

$$\frac{\mathrm{Var}(X)}{k^2} \geq P((X - E(X))^2 \geq k^2)$$

Also by observation 1: $P(|X - E(X)| \geq k) = P((X - E(X))^2 \geq k^2)$. Thus we have:

$$\frac{\mathrm{Var}(X)}{k^2} \geq P|(X - E(X)| \geq k)$$

which completes the argument.

Problem 12: Let X an nonnegative random variable that only takes on integer values. Show that $P(X > 0) \leq E(X)$

By definition of expectation, we have $E(X) = 1P(X = 1) + 2P(X = 2) + \cdots + kP(X = k) + \ldots$. From here, we have the following inequality:

$$E(X) \geq P(X = 1) + P(X = 2) + \cdots + P(X = k) + \ldots$$

Thus $E(X) \geq \sum_{x>0} P(X = x)$, and notice that $\sum_{x>0} P(X = x)$.

Therefore $E(X) \geq P(X > 0)$ as desired.

Problem 13: Let X an nonnegative random variable (not always 0) that only takes on integer values. Show that $P(X > 0) \geq \frac{(E(X))^2}{E(X^2)}$

We utilize the law of total expectation:

$$E(X^2) = E(X^2|X > 0)P(X > 0) + E(X^2|X = 0)P(X = 0)$$
$$E(X^2) = E(X^2|X > 0)P(X > 0)$$

Now by Jensen's inequality we have:

$$E(X^2) = E(X^2|X > 0)P(X > 0) \geq (E(X|X > 0))^2 P(X > 0)$$
$$E(X^2) \geq (E(X|X > 0))^2 P(X > 0)$$

Now we make the following observation, again by the law of total probability:

$$E(X) = E(X|X > 0)P(X > 0) + E(X|X = 0)P(X = 0)$$
$$E(X) = E(X|X > 0)P(X > 0)$$
$$\frac{E(X)}{P(X > 0)} = E(X|X > 0)$$

Therefore:

$$E(X^2) \geq (E(X|X > 0))^2 P(X > 0) = \frac{(E(X))^2}{P(X > 0)}$$
$$E(X^2) \geq \frac{(E(X))^2}{P(X > 0)}$$
$$P(X > 0) \geq \frac{(E(X))^2}{E(X^2)}$$