# UCLA Stats/Prob Challenges

Andrew Lizarraga

**Stats/Prob Challenges** **2025**

# Week 0: Problems

*Author: Andrew Lizarraga*

## About These Problems

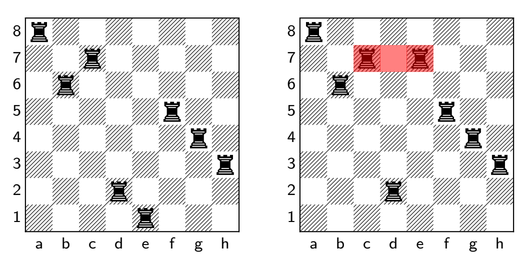
- These problems (and their solutions) will be distributed once a week.
- The problems are mostly probabilistic, however I toss a couple of different topic problems. So keep your guard up.
- They are meant for intellectual entertainment! So have fun and run them by friends.

## Let's Start

**Problem 1 (Coin Flippers):** I flip a fair coin 10 times and they all came up heads. Is it more likely that the $11^{th}$ coin flip will come up tails?

**Problem 2 (Coin Toss):** I flip two fair coins simultaneously, when they hit the ground, what's the probability of getting one coin showing a heads and the other a tails?

**Problem 3 (Peaceful Rooks):** You are given an $8 \times 8$ chess board and you need to place 8 rooks on the boards so that no two rooks can attack each other, i.e. a **peaceful** arrangement. How many peaceful arrangements are there? (Remember rooks can only go left, right, up and down. See below for an example of a peaceful and un-peaceful arrangement, respectively)



**Problem 4 (The UCXB Students):** Two students at UCXB, decided to party before the final. Unfortunately, due to their poor judgement they missed the final. They told the Professor that they were driving back from a families home but one of the tires went flat which is why they missed the final (this is a lie). The Professor agreed to let them take a different final. The two students were seated in separate rooms. There was only one question on the final worth 100 percent of the grade: "Which tire was it?"

What is the probability that the students give the same answer?

**Problem 5 (Aurora Borealis 1):** Friday is going to have a clear night. There is a 60% chance that you can see the aurora borealis in any given hour. If you go outside and watch the sky for two hours, what's the probability that you'll see the aurora borealis?

**Problem 6 (Aurora Borealis 2):** Saturday night is going to be clear as well! Conditions are even better this time, there is an 80% chance of seeing the aurora borealis at any given hour. Let's assume that the probability is uniform for the entire hour. What is the probability you'll see the aurora borealis in the first 15 minutes?

**Problem 7 (A Message?):** Initially, lions often visit elephants, snakes take antelopes, tigers invade stables, toucans irk cheetahs steadily.

**Problem 8 (Anything Else?):** Is there anything that you'd like to see in this course? Anything I should be made aware of? Any concerns you may have? Also, what's your major and year by the way?

**Stats/Prob Challenges** **2025**

# Week 0: Solutions

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## Solutions

**Problem 1 (Coin Flippers):** I flip a fair coin 10 times and they all came up heads. Is it more likely that the $11^{th}$ coin flip will come up tails?

*Solution:* The coin is a fair coin. And it doesn't have any memory about the past flips, it doesn't matter if the previous 10 flips are all tails. The next coin flip is just as likely to be heads as it is to be tails.

**Problem 2 (Coin Toss):** I flip two fair coins simultaneously, when they hit the ground, what's the probability of getting one coin showing a heads and the other a tails?

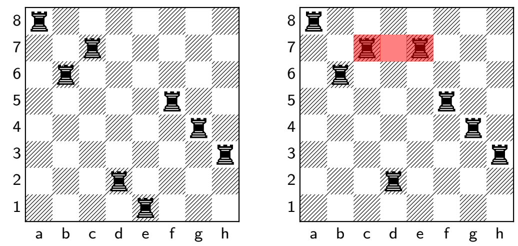
*Solution:* Here are the following outcomes for the two coins: $\{HH, HT, TH, TT\}$. Since we are interested in the $HT$ and the $TH$ outcomes, thus there is a $\frac{2}{4} = \frac{1}{2}$ chance of this occurring.

*Remark:* Notice that we distinguish $HT$ from $TH$, even though we tossed the coins simultaneously. You might think that we shouldn't distinguish these two events because of this. However, it is essential to consider whether it makes sense to treat $HT$ and $TH$ as the same event. The answer is no.

Consider the probability of getting $HH$. There is a $\frac{1}{2}$ chance of getting $H$ for one coin and the same probability for the other. Since the coins are independent, the probability of getting $HH$ is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Likewise the probability for $TT$ is $\frac{1}{4}$. It is an axiom of probability that the total probability of distinct outcomes must add up to 1. Since we consider $HT = TH$, we have $P(HH) + P(TT) + P(HT) = 1$, which implies that $P(HT) = \frac{1}{2}$. However, because the coins are independent, the probability of $HT$ is $P(HT) = P(H)P(T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Thus $HT$ and $TH$ must be treated as separate events.

Another way to understand why we care about the order is to note that when the coins are tossed, there is a first coin that hits the ground, followed by the second coin. Therefore we can order the outcomes: $\{HH, HT, TH, TT\}$ as stated earlier.

**Problem 3 (Peaceful Rooks):** You are given an $8 \times 8$ chess board and you need to place 8 rooks on the boards so that no two rooks can attack each other, i.e. a **peaceful** arrangement. How many peaceful arrangements are there? (Remember rooks can only go left, right, up and down. See below for an example of a peaceful and un-peaceful arrangement, respectively)




*Solution:* Since we have to place 8 rooks on the $8 \times 8$ chessboard, then it must be the case that each row of the chessboard requires exactly one rook. Start with row 1, we have 8 choices to place the rook. Now once the first rook is placed, there are 7 choices to place the second rook in the second row (because we can't place the rook in the column associated with the first rook we placed earlier). Once we place the second rook, we have 6 choices for the third rook to be placed in the third row, and we continue in this fashion until we get to the final row in which we only have one choice to place the rook. Thus there are $8! = 8 \times 7 \times 6 \ldots 3 \times 2 \times 1$ possible peaceful arrangements.

**Problem 4 (The UCXB Students):** Two students at UCXB, decided to party before the final. Unfortunately, due to their poor judgement they missed the final. They told the Professor that they were driving back from a families home but one of the tires went flat which is why they missed the final (this is a lie). The Professor agreed to let them take a different final. The two students were seated in separate rooms. There was only one question on the final worth 100 percent of the grade: "Which tire was it?"

What is the probability that the students give the same answer?

*Solution:* Let's think of the pairs of answers the students can give. They are in separate rooms, so we treat them as being independent and assume that they will pick any tire wih equal probability. We label the tires $1, 2, 3, 4$. There are $4 \times 4 = 16$ possible pairs of answers and only 4 of the pairs, namely $\{11, 22, 33, 44\}$, have both answers being the same, thus the probability of both students giving the same answer is $\frac{4}{16} = \frac{1}{4}$.

*Remark:* This problem can also be solved with conditional probability. The first student will pick a tire, then the probability the second student will pick the same tire is $\frac{1}{4}$ since of the 4 options they have, only one of them matches the first student's answer.

**Problem 5 (Aurora Borealis 1):** Friday is going to have a clear night. There is a 60% chance that you can see the aurora borealis in any given hour. If you go outside and watch the sky for two hours, what's the probability that you'll see the aurora borealis?

*Solution:* Let $S$ denote that we see the auroras in a given hour, and $N$ denote that we didn't see the auroras in a given hour. Then, the possible states for the 2 hour period is $\{SS, SN, NS, NN\}$. Since we only care about the probability of seeing the Auroras, thus we want:
$P(\{SS, SN, NS\}) = P(SS) + P(SN) + P(NS) = P(S)P(S) + P(S)P(N) + P(N)P(S)$

Thus $P(\{SS, SN, NS\}) = (0.6)^2 + (0.6)(0.4) + (0.4)(0.6) = 0.84$
There is an 84% chance of seeing the aurora borealis.

**Problem 6 (Aurora Borealis 2):** Saturday night is going to be clear as well! Conditions are even better this time, there is an 80% chance of seeing the aurora borealis at any given hour. Let's assume that the probability is uniform for the entire hour. What is the probability you'll see the aurora borealis in the first 15 minutes?

*A Common Incorrect Answer:* You must avoid this trap: 15 minutes is $\frac{1}{4}$ of an hour, and there's an 80% chance of seeing the auroras in a given hour. Therefore the probability that you see the auroras in the first 15 minutes is $\frac{1}{4}80\% = 20\%$

*Remark:* The reasoning above can't be right. If we apply our knowledge from **problem 5**: We found that with a probability of 60% of seeing the auroras in a given hour led to a 84% probability of seeing the auroras in two hours. If we use the faulty logic from earlier, we'd conclude that there was only a 42% chance of seeing the auroras in 1 hour, which is clearly wrong.

*Solution:* An easier approach is to instead consider the probability you don't see the auroras for the entire hour. We know that that the probability of seeing the auroras in a given hour is 80%, so the probability that we don't see it in the entire hour is $100\% - 80\% = 20\%$. Equivalently, this occurs if you don't see the auroras for each of the 4-quarter hours that make up the entire hour.
Let $p$ be the probability of seeing the auroras in the first 15 minutes. Also note that since the probability is uniform, the probability that you see the auroras for any particular 15 minute interval is also $p$.
Now we can denote the probability of not seeing the auroras in the first hour by:

$P(\text{None in first hour}) =$

$P(\text{None in first 15 minutes}) \times P(\text{None in second 15 minutes}) \times P(\text{None in third 15 minutes}) \times P(\text{None in fourth 15 minutes})$

$= (1-p)^4$

Thus $(1-p)^4 = 20\% = \frac{1}{5}$. Solving for $p$ yields $p = 1 - \frac{1}{\sqrt[4]{5}} \approx 33.1\%$

**Problem 7 (A Message?):** Initially, lions often visit elephants, snakes take antelopes, tigers invade stables, toucans irk cheetahs steadily.

*Solution:* The keyword is *Initially*, i.e. think about the initials of each word:

**I**nitially, **L**ions **O**ften **V**isit **E**lephants, **S**nakes **T**ake **A**ntelopes, **T**igers **I**nvade **S**tables, **T**oucans **I**rk **C**heetahs **S**teadily.

**I LOVE STATISTICS** Well, at least I hope you do!

**Problem 8 (Anything Else?):** Is there anything that you'd like to see in this course? Anything I should be made aware of? Any concerns you may have? Also, what's your major and year by the way?

*Solution:* This is an exercise left for the reader.

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# Week 1: Problems

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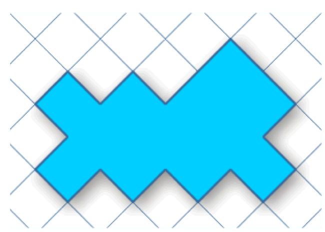
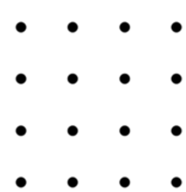
## 1.1 Are you ready?

**Problem 1 (Thinking Outside the Box):** Given the $3 \times 3$ grid below, place your pencil at any point. Then, without lifting your pencil you must draw four straight lines to cross out all the 9 dots.

**Problem 2 (One Cut - Two Shapes):** Given the shape below and using a single curve to cut the shape (the curve can be angled or have as many bends as you wish), cut the shape into two identically shaped pieces.



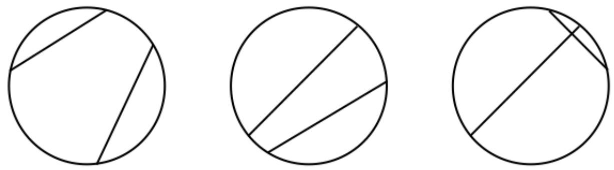
**Problem 3 (Dots and Rectangles):** Given a $4 \times 4$ grid of dots, how many rectangles can be formed by connecting 4 of the dots, such that the sides of rectangle are parallel to the sides of the grid?



**Problem 4 (A Biased Coin):** I give you a coin that is biased in favor of heads. What can you do in order to make it into a fair coin?

**Problem 5 (Half Your Height):** At what age do you expect the average person to be half of their (full-grown) height. Without looking up a statistic on this, can you come up with a plausible way to deduce this answer?

**Problem 6 (A Circle and Two Chords):** I have a unit circle and draw two chords at random. What is the probability that the chords intersect? (Third circle below depicts two chords intersecting).



**Problem 7 (HH vs. HT):** I can choose to flip a fair coin until I see two heads in a row, denoted HH. Or I can choose to flip the coin until I see a heads followed by a tail, i.e. HT. On average should it take more flips to see HH or more flips until you see HT? Or should we expect roughly the same number of flips?

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# Week 1: Solutions

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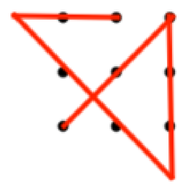
## Solutions

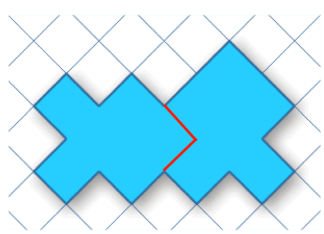
**Problem 1 (Thinking Outside the Box):** Given the $3 \times 3$ grid below, place your pencil at any point. Then, without lifting your pencil you must draw four straight lines to cross out all the 9 dots.

*Solution:* You can draw lines beyond the implicit boundaries of the grid, hence "thinking outside of the box". See the solution below:

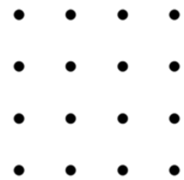


**Problem 2 (One Cut - Two Shapes):** Given the shape below and using a single curve to cut the shape (the curve can be angled or have as many bends as you wish), cut the shape into two identically shaped pieces.

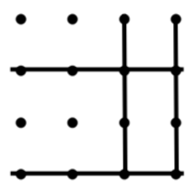
*Solution:* There are many solutions to this problem. Here is one such solution:



**Problem 3 (Dots and Rectangles):** Given a $4 \times 4$ grid of dots, how many rectangles can be formed by connecting 4 of the dots, such that the sides of rectangle are parallel to the sides of the grid?



*Solution:* One possible approach to this problem is to break it down into cases of rectangles of dimensions $1 \times 1$, $1 \times 2$, $2 \times 1$, ..., $4 \times 4$. And then count how many rectangles there are for each case and total them up for you final answer.
While this works it's also time consuming. The astute of you may have made a clever observation. Suppose I placed two horizontal lines on the grid, and then two vertical lines on the grid, then the intersection of these lines will form a rectangle. See below for an example of this:



Thus to determine the total number of rectangles, we need to find the number of ways to place two horizontal lines and two vertical lines on the grid. Notice that there are 4 choose 2 ways to place the horizontal lines, denoted $\binom{4}{2} = 6$. Likewise, there are 6 ways to place the two vertical lines. To reiterate there are 6 ways to place the horizontal lines and then 6 ways to place the vertical lines, for a total of $6 \times 6 = 36$ ways to form a rectangle on the grid.

**Problem 4 (A Biased Coin):** I give you a coin that is biased in favor of heads. What can you do in order to make it into a fair coin?

*Solution:* Let's consider two coin flips and look at the possible states $\{HH, HT, TH, TT\}$. Since the coin is biased in favor of heads, thus $HH$ is most likely and $TT$ is least likely. Also we consider the coin flips to be independent so $TH$ and $HT$ are just as likely. So here is one proposal to make the bias coin fair:

Flip the coin twice. If you get $HH$ or $TT$ disregard the result and flip twice again. Otherwise if you get $HT$ relabel this as $H$ and if you get $TH$ relabel this as $T$. Since $HT$ is just as likely as $TH$, this ensure that our relabeling trick assigns equal probabilities to $H$ and $T$, which makes the coin fair again.

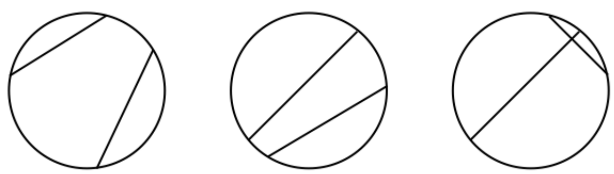
**Problem 5 (Half Your Height):** At what age do you expect the average person to be half of their (full-grown) height. Without looking up a statistic on this, can you come up with a plausible way to deduce this answer?

*Remark:* There isn't really a "correct" way to do this. This problem is more to test if you can make a reasonable conclusion based on your basic prior knowledge about the world.

*Solution:* It's commonly stated that you are fully grown around the age of 18. Let's just say the average height of the typical adult is 6 feet. Well if you were half this height at the age of 9 then you would be 3 feet at the age of 9. This seems off, 3 is more associated with a toddler, say around the age of $9/2 = 4.5 \approx 4$ years old. Now I stated 6 feet was the average height, but realistically that's probably too tall and the average height is maybe closer to 5 feet 6 inches. So 4 years old might still be an estimate, so I would argue you're about half your height around an age between 3 and 4.

*Remark:* The real life statistics is more around the range of 2 to 3.

**Problem 6 (A Circle and Two Chords):** I have a unit circle and draw two chords at random. What is the probability that the chords intersect? (Third circle below depicts two chords intersecting).

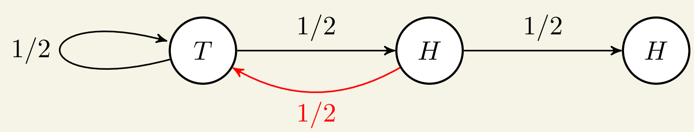


*Solution:* Let's place 4 points on the circle in an arbitrary position. How many ways can you draw the pairs of chords given these 4 points? You should see that there are only 3 ways to do this, one of which depicts two chords intersecting. Thus there is a $\frac{1}{3}$ probability of the chords intersecting. Since the choice of the 4 points was arbitrary, this argument applies for any arrangements of 4 points in general position and thus for the distribution of drawing two chords on the circle. Therefore the probability of two chords intersecting is $\frac{1}{3}$.

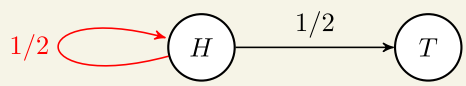
**Problem 7 (HH vs. HT):** I can choose to flip a fair coin until I see two heads in a row, denoted HH. Or I can choose to flip the coin until I see a heads followed by a tail, i.e. HT. On average should it take more flips to see HH or more flips until you see HT? Or should we expect roughly the same number of flips?

*Solution:* This problem is counterintuitive. The answer is that it take more flips on average to see $HH$ than it does $HT$. To get an intuitive understanding consider the following argument:

**Case HH:**
Suppose we just flipped a $H$ (center circle in figure below). Now if we flip a coin and fail to get $H$ (i.e. $T$), then at best we are 2 flips away from obtaining $HH$.




**Case HT:**
Suppose we just flipped a $H$ (left circle in figure below). Now if we flip a coin and fail to get $T$ (i.e. $H$), then at best we are 1 flips away from obtaining $HT$.




So it's more likely that you see $HT$ before you see $HH$ and therefore we expect to see $HT$ in fewer flips than $HH$.

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# Week 2: Problems

*Author: Andrew Lizarraga*

## Welcome back challenger! Ready for more?

**Problem 1 (Bag Of Cards):** A bag contains 3 cards. One card is black on both sides, another card is white on both sides, and the third card is black on one side while being white on the other side. The cards are mixed up in the bag. Then a single card is selected and placed on a table so that the visible side of the card is black. What is the probability of the other side of the card being white?

**Problem 2 (In Between):** I take three samples uniformly from the unit interval $[0, 1]$ and denote them $A, B, C$. What is the probability that $A < B < C$ ?

**Problem 3 (An Obscuring Object?):** A man in a gallery is asked about a portrait depicting a man with an object obscuring his face. He responds, 'Brothers and sisters I have none, but that man's father is my father's son.' What is the object obscuring the face of the man in the portrait?

**Problem 4 (Three Slips):**

Three different numbers are chosen at random, and one is written on each of three slips of paper. The slips are then placed face down on the table. If you choose the slip with the largest number you win!

*Rules:*

1. Initially, you can turn over any slip of paper and view the number written on it.
2. You may choose to keep the initial slip.
3. If unhappy with your first choice, you may discard your it and view another slip. You can choose to keep your second choice or discard it.
4. If you discard the second slip, then you must keep the third slip
5. You win if the slip you keep has the largest number out of all 3 slips.

What is the optimal strategy to increase your odds of winning?

**Stats/Prob Challenges** **2025**

# Week 2: Solutions

*Author: Andrew Lizarraga*

## Solutions

**Problem 1 (Bag Of Cards):** A bag contains 3 cards. One card is black on both sides, another card is white on both sides, and the third card is black on one side while being white on the other side. The cards are mixed up in the bag. Then a single card is selected and placed on a table so that the visible side of the card is black. What is the probability of the other side of the card being white?

*Solution:* Label the faces of the cards, say $b_1, b_2$ for the double black card, $w_1, w_2$ for the double white card, and $b_3, w_3$ for the white and black card. Then the event of seeing a black side of the card on the table corresponds to one of the following evens: $b_1, b_2, b_3$. Only the $b_3$ card corresponds to the white face $w_3$, thus there is a $\frac{1}{3}$ chance that the other side of the card is white.

**Problem 2 (In Between):** I take three samples uniformly from the unit interval $[0, 1]$ and denote them $A, B, C$. What is the probability that $A < B < C$ ?

*Solution:* Let's say the three points $A, B, C$ are chosen uniformly at random on the interval $[0, 1]$. Then there are $3! = 6$ possible orderings of the three unique values and therefore a $\frac{1}{6}$ chance that the numbers drawn or linearly ordered and coincide with $A < B < C$.

**Problem 3 (An Obscuring Object?):** A man in a gallery is asked about a portrait depicting a man with an object obscuring his face. He responds, 'Brothers and sisters I have none, but that man's father is my father's son.' What is the object obscuring the face of the man in the portrait?

*Solution:* Perhaps you are familiar with the original riddle which states *'Brothers and sisters I have none, but that man's father is my father's son.'* The answer to the original riddle is that the man in the portrait is the *son of the man.* If you know some art history or look up the portrait *The Son of Man*, then you'll know that this painting depicts a man with a green apple covering his face. So the the answer is a **green apple**.

## Problem 4 (Three Slips):

Three different numbers are chosen at random, and one is written on each of three slips of paper. The slips are then placed face down on the table. If you choose the slip with the largest number you win!

*Rules:*

1. Initially, you can turn over any slip of paper and view the number written on it.
2. You may choose to keep the initial slip.
3. If unhappy with your first choice, you may discard your it and view another slip. You can choose to keep your second choice or discard it.
4. If you discard the second slip, then you must keep the third slip
5. You win if the slip you keep has the largest number out of all 3 slips.

What is the optimal strategy to increase your odds of winning?

*Solution:* For simplicity, let's just say the 3 randomly chosen numbers are $1, 2, 3$. Then the possibilities that you might face when turning the slips over are $\{123, 132, 213, 231, 312, 321\}$. Here is a proposed strategy, we will always turn two slips over. If the second slip is larger than the first, we keep it. There is $\frac{1}{2}$ chance of this occurring and of these cases $\{123, 132, 231\}$ 2 of the cases are winning.

If the second slip is less than the first slip, we flip the third slip and keep it. Again there is a $\frac{1}{2}$ chance of this occurring and among these cases $\{213, 312, 321\}$ only one is a winning case.

Following this strategy we have 3 winning cases and 3 losing cases, so we have a $\frac{1}{2}$ chance of winning.

Notice that if our strategy was to always pick the first slip, we'd win $\frac{2}{6} = \frac{1}{3} < \frac{1}{2}$. A similar argument holds for always choosing the second or third slip. This only leaves the strategy mentioned earlier in the solution, hence it's optimal.

**Stats/Prob Challenges** **2025**

Week 3: Problems

*Author: Andrew Lizarraga*

## 3.2 Welcome back challenger! I wasn't expecting you ...

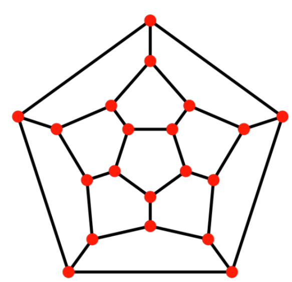
**Problem 1 (Poker Dice):** I have five fair dice, each with the following faces: $9, 10, J, Q, K, A$. When rolling these 5 dices, what's the probability of 2 pairs?

**Problem 2 (The Trapped Miner):** A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after 2 hours of travel. The second door leads to a tunnel that returns him to the mine after 3 hours of travel. The third door leads to a tunnel that returns him to the mine after 5 hours of travel. Assuming the miner at any time is equally likely to choose any door, what is the expected amount of time until the miner reaches safety?

**Problem 3 (The First Ace):** You are given a well-shuffled standard deck of 52 cards. You start at the top of the deck and turn a card over one-at-a-time. What is the expected number of cards that will be turned over before you see the first Ace?

**Problem 4 (Dot Joining):** There are 2024 dots lying in a circle. Suddenly, each dot either forms an edge to a dot on its left or to it right or forms no edge whatsoever (equal chances of each event occurring). About how many isolated dots are there?

**Problem 5 (A Dodecahedral Traversal):** The image below depicts the planar graph representation of a dodecahedron. Start at any point. Can you traverse this graph (meet every vertex not necessarily every edge) without retracing your steps or intersecting your path and end up at the same point?



**Problem 6 (Buffon's Needle):** We have parallel lines that are separated 1-unit away from each other and they span the entire plane.

**Problem 7 (Laplace's Needle):** We have a set of parallel lines that are separated 1-unit away from each other. Additionally, we have another set of parallel lines, orthogonal to the first set of parallel lines, that are also spaced 1-unit apart. So the plane is effectively covered by a grid of unit squares. What is the probability that when tossing a needle that it crosses a line?

**Stats/Prob Challenges** **2025**

Week 3: Solutions

*Author: Andrew Lizarraga*

## Solutions

**Problem 1 (Poker Dice):** I have five fair dice, each with the following faces: $9, 10, J, Q, K, A$. When rolling these 5 dices, what's the probability of 2 pairs?

*Solution:* The total number of possible outcomes is $6^5$. Now we need to consider the number of ways in which there are two pairs. We do this in the following way:

- There are $\binom{6}{2} = 15$ ways to choose 2 ranks for the pairs.
- There is $\binom{4}{1} = 4$ ways to choose 1 rank for the fifth die that is different from the 2 pairs.
- We choose 2 dice out of 5 to be the first pair. There is $\binom{5}{2} = 10$ ways to do this.
- We choose 2 dice out of the remaining 3 for the second pair in $\binom{3}{2} = 3$ ways.

Thus the probability of 2 pairs is given by $\frac{15\times4\times10\times3}{6^5} \approx 0.23$

**Problem 2 (The Trapped Miner):** A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after 2 hours of travel. The second door leads to a tunnel that returns him to the mine after 3 hours of travel. The third door leads to a tunnel that returns him to the mine after 5 hours of travel. Assuming the miner at any time is equally likely to choose any door, what is the expected amount of time until the miner reaches safety?

*Solution:* Let $T$ denote the variable for the total time it takes for the miner to escape. Let $D \in \{1, 2, 3\}$ denote the door the miner chooses. Then by the law of total expectation we have:

$$E[T] = E[T|D = 1]P(D = 1) + E[T|D = 2]P(D = 2) + E[T|D = 3]P(D = 3)$$

$$E[T] = E[T|D = 1]\frac{1}{3} + E[T|D = 2]\frac{1}{3} + E[T|D = 3]\frac{1}{3} \quad (\star)$$

Now observe that $E[T|D = 1] = 2$, $E[T|D = 2] = 3 + E[T]$, and $E[T|D = 3] = 5 + E[T]$. By substituting these terms into $(\star)$, we have:

$$E[T] = \frac{1}{3}(2 + (3 + E[T]) + (5 + E[T]))$$

$$3E[T] = 10 + 2E[T]$$

$$E[T] = 10$$

So we expect it to take about 10 hours for the miner to reach safety.

**Problem 3 (The First Ace):** You are given a well-shuffled standard deck of 52 cards. You start at the top of the deck and turn a card over one-at-a-time. What is the expected number of cards that will be turned over before you see the first Ace?

*Solution:* You can think of the 4 Aces as dividing the other 48 cards of the deck into 5 groups: The cards before the first Ace, the cards between the first and second Ace, and so on. The expected size of each group is $\frac{48}{5} = 9.6$. Thus we need to turn all of the cards over in the first group, plus the the next card for the first Ace. So the expected number of cards we need to turn over is $1 + 9.6 = 10.6$.

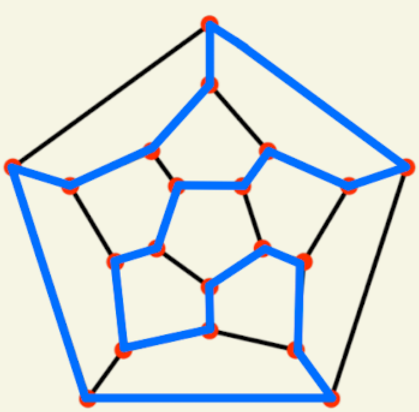
**Problem 4 (Dot Joining):** There are 2024 dots lying in a circle. Suddenly, each dot either forms an edge to a dot on its left or to it right or forms no edge whatsoever (equal chances of each event occurring). About how many isolated dots are there?

*Solution:* Let $X_i$ be a random variable associated to the $i$-th dot on the circle. We let $X_i = 1$ if the dot is isolated, otherwise $X_i = 0$. Note, there is a $\frac{1}{3}$ probability it joins an arc to the dot on it's left, a $\frac{1}{3}$ probability it joins an arc to a dot on it's right, and a $\frac{1}{3}$ probability that it remains isolated. Thus, $X_i \sim Ber(p = \frac{1}{3})$. So the expected number of isolated points is given by:

$$E(X_1 + \cdots X_{2024}) = \sum_{i=1}^{2024} \frac{1}{3} = \frac{2024}{3} \approx 674.76$$

**Problem 5 (A Dodecahedral Traversal):** The image below depicts the planar graph representation of a dodecahedron. Start at any point. Can you traverse this graph (meet every vertex not necessarily every edge) without retracing your steps or intersecting your path and end up at the same point?

*Solution:* Consider the image below. Starting at the top vertex in the image and traversing the blue path counterclockwise is a solution.



**Problem 6 (Buffon's Needle):** We have parallel lines that are separated 1-unit away from each other and they span the entire plane. We toss a needle of length 1 onto the plane, what is the probability that the needle crosses a line?

*Solution:* The key observation is determining a good representation of the needle. Before we do that let's make a couple of observations:

1. With probability 1 the center of the needle must land in some parallel strip. So it's enough to focus on a single strip **(Why?)**.

2. More so, within the strip that contains the center of the needle, we also know with probability 1 that the needle must be on one-half of the strip **(Why?)**.

So it's enough to solve the problem in this case. We denote the distance from the center of the needle to the nearest parallel line by $x$, and denote the acute angle formed by the needle and the orthogonal projection of the center of the needle and the parallel line, by $\theta$.

So the position of the needle is completely captured by $(x, \theta)$ where $x \sim U[0, \frac{1}{2}]$ and $\theta \sim U[0, \frac{\pi}{2}]$. So we may think of the sample space as $\Omega \equiv [0, \frac{1}{2}] \times [0, \frac{\pi}{2}]$. Now we must determine what region corresponds to the needle crossing the parallel line. Let's call this region $A$. Once we find this region, then we can determine the probability of a needle crossing a parallel line by computing $\frac{|A|}{|\Omega|}$.

Notice that we can relate $x$, $\theta$, and hypotenuse (half of the needle) via $\cos\theta = \frac{x}{2} = 2x$. Thus $x = \frac{\cos\theta}{2}$. Notice that if $\theta = 0$, then the needle is orthogonal to the parallel line on the plane, so any $x$ in the range from 0 up to $\frac{1}{2}$ indicate the needle is intersecting the parallel line. Also since $\frac{\cos\theta}{2}$ is a concave function, thus we can conclude that $x \leq \frac{\cos\theta}{2}$ is the region corresponding to the needle crossing.

As mentioned early, the probability corresponds to computing $\frac{|A|}{|\Omega|}$. Since $\Omega \equiv [0, \frac{1}{2}] \times [0, \frac{\pi}{2}]$, thus $|\Omega| = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$. To compute $|A|$ corresponds computing the area under the curve $x = \frac{\cos\theta}{2}$ which is given by

$$|A| = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{2} d\theta = \frac{1}{2}$$

Therefore the probability of a needle crossing is $\frac{|A|}{|\Omega|} = \frac{2}{\pi}$.

Now to run a Monte Carlo Simulation, we can note the following. Toss $n$ needles, and suppose that $m$ of them cross a parallel line. Then by what we've shown, we expect that $\frac{m}{n} \to \frac{2}{\pi}$. This can be reexpressed as: $\frac{2n}{m} \to \pi$. So by computing the fraction $\frac{2n}{m}$, we are computing an approximation of $\pi$.

**Problem 7 (Laplace's Needle):** We have a set of parallel lines that are separated 1-unit away from each other. Additionally, we have another set of parallel lines, orthogonal to the first set of parallel lines, that are also spaced 1-unit apart. So the plane is effectively covered by a grid of unit squares. What is the probability that when tossing a needle that it crosses a line?

*Solution:* Consider the center of the needle. We know with probability 1 that it lands in some unit square. Now, we consider how to parameterize the needle. One intuitive parameterization is letting $(x, y)$ denote the center of the needle. By a similar argument as the Buffon Needle problem, we need only focus on the sample space given by with $x \sim U[0, \frac{1}{2}]$ and $y \sim U[0, \frac{1}{2}]$. We can let $\theta$ be the acute angle formed by the needle and the $x$-axis, so $\theta \sim U[0, \frac{\pi}{2}]$.

So our sample space consists of points, $(x, y, \theta)$, belonging to the cube $\Omega \equiv [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, \frac{\pi}{2}]$. Notice that the volume of the cube is given by $|\Omega| = \frac{\pi}{8}$. So to determine the probability of a needle crossing a grid-line in Laplace's experiment, it suffices to compute the volume of the region $A \subset \Omega$ that corresponds to the needle crossing and then dividing this volume by the total volume, $|\Omega|$.

We let $I_A(x, y, \theta)$ denote the indicator function, which evaluates to 1 when the tuple $(x, y, \theta)$ corresponds to values indicating the needle has crossed a grid-line in the plane (otherwise it is 0). So the volume of $A$ is given by $|A| = \iiint_\Omega I_A(x, y, \theta) dxdyd\theta$. Recall from Buffon's experiment that a condition for the needle to cross a parallel line with respect to $x$ is given by $x \leq \frac{\cos\theta}{2}$. Similarly, by use of the complementary angle, we have that the needle crossing condition with respect to $y$ is given by $y \leq \frac{\cos(\frac{\pi}{2} - \theta)}{2} = \frac{\sin\theta}{2}$. Keeping in mind that it's possible for the needle to cross orthogonal lines. Thus we may express the volume $|A|$ as follows:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} I\left(x \leq \frac{\cos\theta}{2}\right) + I\left(y \leq \frac{\sin\theta}{2}\right) - I\left(x \leq \frac{\cos\theta}{2}\right) I\left(y \leq \frac{\sin\theta}{2}\right) dxdyd\theta$$

where $I(\cdot)$ indicates on the region where the respective condition is true. We may solve the integral in it's three terms:

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} I\left(x \leq \frac{\cos\theta}{2}\right) dxdyd\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{\cos\theta}{2}} dxdyd\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \frac{\cos\theta}{2} dyd\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{4} d\theta \\
&= \frac{1}{4}
\end{aligned}$$

Similarly,

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} I\left(y \leq \frac{\sin\theta}{2}\right) dx dy d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\sin\theta}{2}} \int_0^{\frac{1}{2}} dx dy d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\sin\theta}{2}} \frac{1}{2} dy d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{4} d\theta$$

$$= \frac{1}{4}$$

Lastly,

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} -I\left(x \leq \frac{\cos\theta}{2}\right) I\left(y \leq \frac{\sin\theta}{2}\right) dx dy d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\sin\theta}{2}} \int_0^{\frac{\cos\theta}{2}} -1 \ dx dy d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\sin\theta}{2}} -\frac{\cos\theta}{2} dy d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{\cos\theta\sin\theta}{4} d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{\sin 2\theta}{8} d\theta$$

$$= -\frac{1}{8}$$

Therefore $|A| = \frac{1}{4} + \frac{1}{4} - \frac{1}{8} = \frac{3}{8}$. Now recall that $|\Omega| = \frac{\pi}{8}$, thus the probability of a needle intersecting is given by $\frac{|A|}{|\Omega|} = \frac{3}{\pi}$. Again, among $n$ tosses if we count $m$ the number of needle crosses, then we expect $\frac{m}{n} \to \frac{3}{\pi}$ as $n$ gets large. If we ran Laplace's experiment, then we expect $\frac{3n}{m} \to \pi$.

# Week 4: Problems

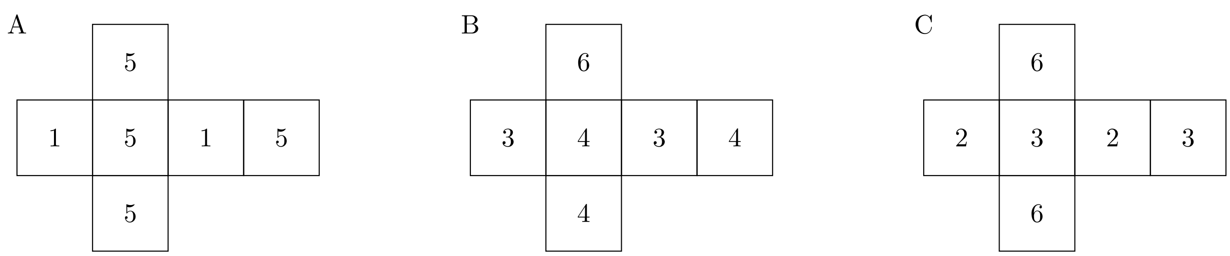
*Author: Andrew Lizarraga*

## 4.3 Welcome back challenger! I wait for you conditionally so.

**Problem 1 (Repulsive Probability):** The event $A$ is said to be repelled by the event $B$ if $P(A|B) < P(A)$, and it is said to be attracted to $B$ if $P(A|B) > P(A)$. Show that if $B$ attracts $A$, then $A$ attracts $B$, and $B^c$ repels A. If $A$ attracts $B$, and $B$ attracts $C$, does $A$ attract $C$?

**Problem 2 (Around The Quarter):** A quarter is glued to a tabletop. You then place another quarter tangent to the tabletop quarter and begin rotating it around the tabletop quarter. You keep rotating it until your quarter reaches the point it started at. How many times will your quarter rotate?

**Problem 3 (Rigged Dice):** I have dice, $A, B, C$ with face values depicted below:




Show that $P(A > B) = P(B > C)$. Is it the case that $P(A > C) = P(A > B) = P(B > C)$?

**Problem 4 (Coin Game I):** Player $A$ has 100 fair coins and player $B$ has 101 fair coins. Both of them toss their respective coins simultaneously and count the number of heads they each respectively received. Whoever has more heads is declared the winner winner. What's the probability that player $B$ beats player $A$.

**Problem 5 (Coin Game II):** Players $A$ and $B$ are playing a game where they take turns flipping a biased coin, with probability $p$ of landing heads (and winning). Player $A$ starts the game, and then the players pass the coin back and forth until one person flips heads and wins.

What is the probability that $A$ wins?

**Stats/Prob Challenges** **2025**

# Week 4: Solutions

*Author: Andrew Lizarraga*

## Solutions

**Problem 1 (Repulsive Probability):** The event $A$ is said to be repelled by the event $B$ if $P(A|B) < P(A)$, and it is said to be attracted to $B$ if $P(A|B) > P(A)$. Show that if $B$ attracts $A$, then $A$ attracts $B$, and $B^c$ repels A. If $A$ attracts $B$, and $B$ attracts $C$, does $A$ attract $C$?

*Solution:* Suppose $A$ attracts $B$, then by definition we have that $P(B|A) > P(B)$. Also recall that $P(B|A) = \frac{P(A \cap B)}{P(A)}$. So we have that $\frac{P(A \cap B)}{P(A)} > P(B)$, and multiplying both sides of the inequality by $\frac{P(A)}{P(B)}$, we have $\frac{P(A \cap B)}{P(B)} > P(A)$. Therefore $P(A|B) > P(A)$, which proves that $A$ attracts $B$. To show that $B^c$ repels $A$, notice that by the total law of probability that $P(A) = P(A|B^c)P(B^c) + P(A|B)P(B)$, and since $B$ attracts $A$, we have the inequality:

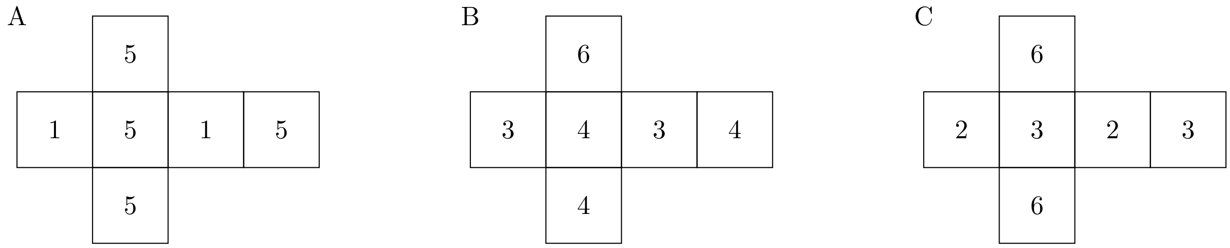
$$P(A) > P(A|B^c)P(B^c) + P(A)P(B)$$
$$P(A)(1 - P(B)) > P(A|B^c)P(B^c)$$
$$P(A)P(B^c) > P(A|B^c)P(B^c)$$
$$P(A) > P(A|B^c)$$

Therefore $A$ is repelled by $B^c$.
Lastly, the transitive property doesn't hold. Suppose $A \cap C = \emptyset$, then $P(A|C) = 0 \leq P(A)$.

**Problem 2 (Around The Quarter):** A quarter is glued to a tabletop. You then place another quarter tangent to the tabletop quarter and begin rotating it around the tabletop quarter. You keep rotating it until your quarter reaches the point it started at. How many times will your quarter rotate?

*Solution:* The coin rotates twice.Once relative to the stationary quarter, and once more with respect to it's own revolution.

**Problem 3 (Rigged Dice):** I have dice, $A, B, C$ with face values depicted below:




Show that $P(A > B) = P(B > C)$. Is it the case that $P(A > C) = P(A > B) = P(B > C)$?

*Solution:* Again, we can solve this problem via the law of total probability. Consider the following:

$$\begin{aligned} P(A > B) &= P(A > B|A = 5)P(A = 5) + P(A > B|A = 1)P(A = 1) \\ &= \left(\frac{5}{6}\right)\left(\frac{4}{6}\right) + (0)\left(\frac{2}{6}\right) \\ &= \frac{20}{36} \end{aligned}$$

$$\begin{aligned} P(B > C) &= P(B > C|B = 3)P(B = 3) + P(B > C|B = 4)P(B = 4) + P(B > C|B = 6)P(B = 6) \\ &= \left(\frac{2}{6}\right)\left(\frac{2}{6}\right) + \left(\frac{4}{6}\right)\left(\frac{3}{6}\right) + \left(\frac{4}{6}\right)\left(\frac{1}{6}\right) \\ &= \frac{20}{36} \end{aligned}$$

$$\begin{aligned} P(A > C) &= P(A > C|A = 5)P(A = 5) + P(A > C|A = 1)P(A = 1) \\ &= \left(\frac{4}{6}\right)\left(\frac{4}{6}\right) + (0)\left(\frac{2}{6}\right) \\ &= \frac{16}{36} \end{aligned}$$

Thus $P(A > B) = P(B > C)$ however $P(A > C) \neq P(A > B)$ and $P(A > C) \neq P(B > C)$.

**Problem 4 (Coin Game I):** Player $A$ has 100 fair coins and player $B$ has 101 fair coins. Both of them toss their respective coins simultaneously and count the number of heads they each respectively received. Whoever has more heads is declared the winner winner. What's the probability that player $B$ beats player $A$.

*Solution:* Say we compared 100 coins of players $B$ to player $A$'s 100 coins. Since all these coins are fair, the expectation of number of heads for each player is the same for their respective 100 coins. However, player $B$ has an additional coin, which can be considered a tie-breaker coin, and so with a $\frac{1}{2}$ probability the expected number of heads for $B$ is 1 plus the expected number of heads for $A$. In other words, there is a $\frac{1}{2}$ player $B$ wins.

**Problem 5 (Coin Game II):** Players $A$ and $B$ are playing a game where they take turns flipping a biased coin, with probability $p$ of landing heads (and winning). Player $A$ starts the game, and then the players pass the coin back and forth until one person flips heads and wins.

What is the probability that $A$ wins?

*Solution:* Denote $P(A)$ denote the probability that $A$ wins. There is $p$ chance that player $A$ wins on the first flip. If the first flip is tails (a $1-p$ chance), then the probability of winning is equal to the probability that $B$ gets a tails ($1-p$ chance) and $A$ wins (which is still a $P(A)$ chance by independence). This can also be expressed by the law of total probability:

$$P(A) = P(A|H)P(H) + P(A|T)P(T)$$
$$P(A) = P(A|H)P(H) + (P(T)P(A))P(T)$$
$$P(A) = P(H) + (P(T)P(A))P(T)$$
$$P(A) = p + (1-p)^2 P(A)$$
$$P(A)(1-(1-p)^2) = p$$
$$P(A)(1-1+2p-p^2) = p$$
$$P(A)(2p-p^2) = p$$
$$P(A)(2-p)p = p$$
$$P(A) = \frac{1}{2-p}$$

# Week 5: Problems

*Author: Andrew Lizarraga*

## Okay Challenger, you should review basic Expectation & Variance.

**Problem 1:** I flip a fair coin 10 times. What is the expected number of heads?

**Problem 2:** How many times would I expect to flip a fair coin until I see a heads?

**Problem 3:** How many times would I expect to roll a die until I see a 5?

**Problem 4:** $X$ is a discrete random variable with distribution $q(x)$ and assumes values from $a$ up to $a+n$. What is it's expectation?

**Problem 5:** $X$ is a continuous random variable with distribution $q(x)$, with $q(x) > 0$ for $x \in [a, b]$, otherwise it's 0. What is the expectation of $X$ ?

**Problem 6:** Given a random variable X, what is it's variance? Can you express the variance in two different ways?

**Problem 7:** Is it the case that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ ?

**Problem 8:** Does $E(XY) = E(X)E(Y)$ ?

**Problem 9:** I roll a fair 6-sided die once. Whatever value it lands on, call it $a$. Now roll $a$ dice and take the sum of the face values rolled and call it $b$. What is $E(b)$ ?

**Problem 10:** Let $X$ be a nonnegative integer-valued random variable and $k$ a nonnegative constant. Show that $P(X \geq k) \leq \frac{E(X)}{k}$.

**Problem 11:** Let $X$ be a nonnegative random variable and $k$ a nonnegative constant. Show that $P(|X - E(X)| \geq k) \leq \frac{\text{Var}(X)}{k^2}$

**Problem 12:** Let $X$ an nonnegative random variable that only takes on integer values. Show that $P(X > 0) \leq E(X)$

**Problem 13:** Let $X$ an nonnegative random variable (not always 0) that only takes on integer values. Show that $P(X > 0) \geq \frac{(E(X))^2}{E(X^2)}$

**Stats/Prob Challenges** **2025**

# Week 5: Solutions

*Author: Andrew Lizarraga*

## Solutions

**Problem 1:** I flip a fair coin 10 times. What is the expected number of heads?

*Solution:* Let $X_i$ denote the $i$th flip, where $X_i = 1$ with probability $p = \frac{1}{2}$ and 0 with probability $1 - p = \frac{1}{2}$. Thus by linearity of expectation, we have:

$$E(X_1 + \cdots + X_{10}) = E(X_1) + \cdots + E(X_{10}) = \frac{10}{2} = 5$$

**Problem 2:** How many times would I expect to flip a fair coin until I see a heads?

*Solution:* Denote this random variable by $X$. Notice that $X$ is a geometric random variable, thus the expectation is given by:

$$E(X) = \frac{1}{p} = \frac{1}{1/2} = 2$$

*Remark:* We can also deduce the expectation this way. Recall by the law of total expectation that we have:

$$E(X) = E(X|X = 1)P(X = 1) + E(X|X = 0)P(X = 0)$$
$$E(X) = \frac{1}{2} + (1 + E(X))\frac{1}{2}$$
$$2E(X) = 2 + E(X)$$
$$E(X) = 2$$

**Problem 3:** How many times would I expect to roll a die until I see a 5?

*Solution:* We solve this similarly to problem 2:

$$E(X) = \frac{1}{1/6} = 6$$

**Problem 4:** $X$ is a discrete random variable with distribution $q(x)$ and assumes values from $a$ up to $a+n$. What is it's expectation?

*Solution:* By definition:

$$E(X) = \sum_{x=a}^{a+n} xq(x)$$

**Problem 5:** $X$ is a continuous random variable with distribution $q(x)$, with $q(x) > 0$ for $x \in [a, b]$, otherwise it's 0. What is the expectation of $X$ ?

*Solution:* By definition:

$$E(X) = \int_a^b xq(x)dx$$

**Problem 6:** Given a random variable X, what is it's variance? Can you express the variance in two different ways?

*Solution:* By definition:

$$\text{Var}(X) = E[(X - E[X])^2]$$

Which can also be expressed as:

$$\begin{aligned} &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - 2E[X]^2 + E[X]^2 \\ &= E[X^2] - E[X]^2 \end{aligned}$$

**Problem 7:** Is it the case that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ ?

*Solution:* In short No, unless $X$ and $Y$ are uncorrelated. To properly express the variance, consider the following computation:

$$\begin{aligned} \text{Var}(X + Y) &= E((X + Y)^2) - (E(X + Y))^2 \\ &= E(X^2 + 2XY + Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 \\ &= [E(X^2) - E(X)^2] + [E(Y^2) - E(Y)^2] - 2E(X)E(Y) + 2E(XY) \\ &= \text{Var}(X) + \text{Var}(Y) + 2[E(XY) - E(X)E(Y)] \end{aligned}$$

**Problem 8:** Does $E(XY) = E(X)E(Y)$ ?

*Solution:* No. Suppose $X = 1$ with probability $\frac{1}{2}$ and $X = -1$ with probability $\frac{1}{2}$. Also suppose that $Y = \frac{1}{X}$.
Then $E(XY) = 1$, but $E(X) = 0$ and $E(Y) = 0$, so $E(X)E(Y) = 0$, thus $E(XY) \neq E(X)E(Y)$.

**Problem 9:** I roll a fair 6-sided die once. Whatever value it lands on, call it $a$. Now roll $a$ dice and take the sum of the face values rolled and call it $b$. What is $E(b)$ ?

*Solution:* Note that the expectation of a single fair die roll is $a = \frac{1+2+3+4+5+6}{6} = 3.5$. Now we roll 3.5 more dice on average and each of these dies have an expectation of 3.5 as well. Thus $b = (3.5)(3.5) = 12.25$.

**Problem 10:** Let $X$ be a nonnegative integer-valued random variable and $k$ a nonnegative constant. Show that $P(X \geq k) \leq \frac{E(X)}{k}$.

*Solution:* By definition of expectation we have:

$$E(X) = \sum_{x} xP(X = x)$$

And this can be expressed as:

$$E(X) = \sum_{x<k} xP(X = x) + \sum_{x \geq k} xP(X = x)$$

Notice that since x is nonnegative that:

$$E(X) \geq \sum_{x \geq k} xP(X = x)$$

$$E(X) \geq \sum_{x \geq k} xP(X = x) \geq \sum_{x \geq k} kP(X = x)$$

$$E(X) \geq k \sum_{x \geq k} P(X = x)$$

$$E(X) \geq kP(X \geq k)$$

$$\frac{E(X)}{k} \geq P(X \geq k)$$

**Problem 11:** Let $X$ be a nonnegative random variable and $k$ a nonnegative constant. Show that $P(|X - E(X)| \geq k) \leq \frac{\text{Var}(X)}{k^2}$

*Solution:*
First let's make the following observations:

1. Notice that $P(|X - E(X)| \geq k) = P((X - E(X))^2 \geq k^2)$.
2. If we let $Z = (X - E(X))^2$, then $Z$ is a nonnegative random variable.
3. From the law of total expectation we have:

$$E(Z) = E(Z|Z \geq k^2)P(Z \geq k^2) + E(Z|Z < k^2)P(Z < k^2)$$

Notice that $E(Z|Z \geq k^2) \geq k^2$ since we are conditioning on the fact that $Z \geq k^2$ so the expectation can't be less than $k^2$. Also since $Z$ is nonnegative, we know that $E(Z|Z < k)P(Z < k)$ is also nonnegative and can be set to 0 in order to establish the inequality. So we have:

$$E(Z) \geq k^2 P(Z \geq k^2) + 0$$

$$\frac{E(Z)}{k^2} \geq P(Z \geq k^2)$$

Now substitute $Z = (X - E(X))^2$

$$\frac{E((X - E(X))^2)}{k^2} \geq P((X - E(X))^2 \geq k^2)$$

$$\frac{\text{Var}(X)}{k^2} \geq P((X - E(X))^2 \geq k^2)$$

Also by observation 1: $P(|X - E(X)| \geq k) = P((X - E(X))^2 \geq k^2)$. Thus we have:

$$\frac{\text{Var}(X)}{k^2} \geq P|(X - E(X)| \geq k)$$

which completes the argument.

**Problem 12:** Let $X$ an nonnegative random variable that only takes on integer values. Show that $P(X > 0) \leq E(X)$

By definition of expectation, we have $E(X) = 1P(X = 1) + 2P(X = 2) + \cdots + kP(X = k) + \ldots$.
From here, we have the following inequality:

$$E(X) \geq P(X = 1) + P(X = 2) + \cdots + P(X = k) + \ldots$$

Thus $E(X) \geq \sum_{x>0} P(X = x)$, and notice that $\sum_{x>0} P(X = x)$.

Therefore $E(X) \geq P(X > 0)$ as desired.

**Problem 13:** Let $X$ an nonnegative random variable (not always 0) that only takes on integer values. Show that $P(X > 0) \geq \frac{(E(X))^2}{E(X^2)}$

We utilize the law of total expectation:

$$E(X^2) = E(X^2|X > 0)P(X > 0) + E(X^2|X = 0)P(X = 0)$$
$$E(X^2) = E(X^2|X > 0)P(X > 0)$$

Now by Jensen's inequality we have:

$$E(X^2) = E(X^2|X > 0)P(X > 0) \geq (E(X|X > 0))^2 P(X > 0)$$

$$E(X^2) \geq (E(X|X > 0))^2 P(X > 0)$$

Now we make the following observation, again by the law of total probability:

$$E(X) = E(X|X > 0)P(X > 0) + E(X|X = 0)P(X = 0)$$

$$E(X) = E(X|X > 0)P(X > 0)$$
$$\frac{E(X)}{P(X > 0)} = E(X|X > 0)$$

Therefore:

$$E(X^2) \geq (E(X|X > 0))^2 P(X > 0) = \frac{(E(X))^2}{P(X > 0)}$$

$$E(X^2) \geq \frac{(E(X))^2}{P(X > 0)}$$

$$P(X > 0) \geq \frac{(E(X))^2}{E(X^2)}$$

**Stats/Prob** **2025**

# Week 6: Problems

*Author: Andrew Lizarraga*

## Challenger! Remember the basics of Linear Regression?

We are given that $X \sim N(0,1)$ and that $Y$ is linear function of $X$ given by $Y = \rho X + \varepsilon$, where $|\rho| < 1$ and $\varepsilon \sim N(0, 1-\rho^2)$. We are also given that $\varepsilon \perp X$.

**Problem 1:** Calculate $E[Y|X=x]$.

**Problem 2:** Calculate $\text{Var}[Y|X=x]$.

**Problem 3:** Calculate the joint density $f(x,y)$ based on the chain rule $f(x,y) = f(x)f(y|x)$.

**Problem 4:** Calculate $E[Y]$

**Problem 5:** Calculate $\text{Var}[Y]$

**Problem 6:** Calculate $\text{Cov}(X,Y)$

## Also don't forget your basic Transformation of Random Variables!

Let $U \sim Unif[0,1]$ and let $X = -\log U$.

**Problem 1:** Calculate the cumulative density function $F(x) = P(X \leq x)$.

**Problem 2:** Calculate the probability density function $f(x) = F'(x)$.

**Stats/Prob Challenges** **2025**

# Week 6: Solutions

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## Solutions

We are given that $X \sim N(0,1)$ and that $Y$ is linear function of $X$ given by $Y = \rho X + \varepsilon$, where $|\rho| < 1$ and $\varepsilon \sim N(0, 1-\rho^2)$. We are also given that $\varepsilon \perp X$.

**Problem 1:** Calculate $E[Y|X=x]$.

*Solution:* Notice that by definition of $Y$ and conditional expectation, we have:

$$E[Y|X=x] = E[\rho X + \varepsilon|X=x] = \rho x + E[\varepsilon|X=x]$$

Now since $\varepsilon \perp X$, we have that $E[\varepsilon|X=x] = E[\varepsilon] = 0$. Therefore:

$$E[Y|X=x] = \rho x$$

**Problem 2:** Calculate $\text{Var}[Y|X=x]$.

*Solution:* Again to deduce this result, we use the fact that $Y$ is linear in $X$ and properties of variance and independence:

$$\text{Var}[Y|X=x] = \text{Var}[\rho X + \varepsilon|X=x]$$

Now since $X \perp \epsilon$:

$$\text{Var}[Y|X=x] = \text{Var}[\rho X|X=x] + \text{Var}[\varepsilon|X=x]$$

Note that since $\rho X$ is deterministic given $X$, we have: $\text{Var}[\rho X|X=x] = 0$. Additionally, since $X \perp \varepsilon$, we have $\text{Var}[\varepsilon|X=x] = \text{Var}[\varepsilon] = 1 - \rho^2$.

**Problem 3:** Calculate the joint density $f(x, y)$ based on the chain rule $f(x, y) = f(x)f(y|x)$.

*Solution:* Recall that for $X \sim N(0, 1)$ we have:

$$f(X) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

Also since Normal distributions have a closed form (this is from the lectures), and we computed the mean and variance of $[Y|X = x]$from problems 1 and 2, thus $[Y|X = x] \sim N(\rho x, 1 - \rho^2)$ and,

$$f(y|x) = \frac{1}{\sqrt{2\pi(1 - \rho^2)}} e^{\frac{-(y-\rho x)^2}{2(1-\rho^2)}}$$

Now by the chain rule we have:

$$f(x, y) = f(x)f(y|x) = \left( \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \right) \left( \frac{1}{\sqrt{2\pi(1 - \rho^2)}} e^{\frac{-(y-\rho x)^2}{2(1-\rho^2)}} \right)$$

Simplifying yields:

$$f(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} e^{-\frac{x^2}{2} - \frac{(y-\rho x)^2}{2(1-\rho^2)}}$$

**Problem 4:** Calculate $E[Y]$

*Solution:* The calculation is direct:

$$E[Y] = E[\rho X + \varepsilon] = \rho E[X] + E[\varepsilon]$$

Now since $E[X] = 0$ and $E[\varepsilon] = 0$, thus $E[Y] = 0$.

**Problem 5:** Calculate $\text{Var}[Y]$

*Solution:* The calculation is direct and follows from definition of $Y$, variance and independence of $X$ and $\varepsilon$:

$$\text{Var}[Y] = \text{Var}[\rho X + \varepsilon] = \rho^2 \text{Var}[X] + \text{Var}[\varepsilon]$$

Since $\text{Var}[X] = 1$ and $\text{Var}[\epsilon] = 1 - \rho^2$, therefore:

$$\text{Var}[Y] = \rho^2 + (1 - \rho^2) = 1$$

**Problem 6:** Calculate $\mathrm{Cov}(X, Y)$

*Solution:* Using the following formula from lecture:

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

Now we've have $E[X] = 0, E[Y] = 0$, thus

$$\mathrm{Cov}(X, Y) = E[XY]$$

Since $Y = \rho X + \varepsilon$, thus $XY = \rho X^2 + X\varepsilon$. So we have that

$$E[XY] = E[\rho X^2] + E[X\varepsilon]$$

Since $X \perp \varepsilon$, thus $E[X\varepsilon] = E[X]E[\varepsilon] = 0$. Now we have:

$$E[XY] = \rho E[X^2]$$

Also, since $X \sim N(0, 1)$ we have $E[X^2] = \mathrm{Var}[X] = 1$, therefore:

$$\mathrm{Cov}(X, Y) = E[XY] = \rho$$

Let $U \sim Unif[0,1]$ and let $X = -\log U$.

**Problem 1:** Calculate the cumulative density function $F(x) = P(X \leq x)$.

*Solution:* This follows directly by definition of $X$ and monotonicity:

$$F(x) = P(X \leq x)$$

Now by definition of $X$:

$$= P(-\log U \leq x)$$
$$= P(\log U \geq -x)$$

Since the exponential is monotonic and convex:

$$= P(U \geq e^{-x})$$
$$= 1 - P(U \leq e^{-x})$$

Since $U$ is a uniform random variable the probability that it's bounded between 0 and $e^{-x}$ is just the length of this interval, namely: $P(U \leq e^{-x}) = e^{-x}$. Therefore:

$$F(x) = 1 - e^{-x}$$

**Problem 2:** Calculate the probability density function $f(x) = F'(x)$.

*Solution:*
Since $F(x) = 1 - e^{-x}$, we have that $F'(x) = e^{-x}$.

# Week 7: Problems

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## 7.4 Alright Challenger! The next trials will vary ...

**Problem 1 (A Blood Test):** Suppose that a laboratory test on a blood sample yields one of two results: positive or negative. It is known that 95% of people with a particular disease produce a positive result. But 2% of people without the disease will also produce a positive result (a false positive). Suppose that 1% of the population has the disease. What is the probability that a person chosen at random from the population will have the disease, given that the person's blood yields a positive result.

**Problem 2 (Biased Runs):** A biased coin is tossed $n$ times, and heads shows with probability $p$ on each toss. A run is a sequence of throws which result in the same outcome, so that, for example, the sequence $HHTHTTH$ contains five runs. Show that the expected number of runs is $1 + 2(n-1)p(1-p)$. Find the variance of the number of runs.

**Problem 3 (Urn Removal):** An urn contains $n$ balls numbered $1, \ldots, n$. We remove $k$ balls at random (without replacement) and add up their numbers. Find the mean and variance of the total.

**Problem 4 (Simpson's Game):** You have two bags, each with a mixture of green and red marbles. You can choose a bag, then randomly pick a marble from the bag. If it's green you win $100, but if it's red, you get nothing.

**Game 1:** Bag $A$ contains 5 green marbles and 6 red marbles. Bag $B$ has 3 green marbles and 4 red marbles. What bag should you choose to increase your odds of winning?

**Game 2:** Bag $C$ contains 6 green marbles and 3 red marbles. Bag $D$ has 9 green marbles and 5 red marbles. What bag should you choose to increase your odds of winning?

**Game 3:** We combine bags $A$ and $C$ into bag $E$ and we combine bags $B$ and $D$ into bag $F$. What bag should you choose to increase your odds of winning?

**Stats/Prob Challenges** **2025**

# Week 7: Solutions

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## Solutions

**Problem 1 (A Blood Test):** Suppose that a laboratory test on a blood sample yields one of two results: positive or negative. It is known that 95% of people with a particular disease produce a positive result. But 2% of people without the disease will also produce a positive result (a false positive). Suppose that 1% of the population has the disease. What is the probability that a person chosen at random from the population will have the disease, given that the person's blood yields a positive result.

*Solution:* Let $D$ indicate a person has the disease, let $+$ denote a positive test, and $-$ denote a negative test. We are given $P(+|D) = \frac{95}{100}, P(+|D^c) = \frac{2}{100}, P(D) = \frac{1}{100}$. We are trying to find $P(D|+)$. This is a simple application of Baye's rule:

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)}$$

$$P(D|+) \approx 32\%$$

**Problem 2 (Biased Runs):** A biased coin is tossed $n$ times, and heads shows with probability $p$ on each toss. A run is a sequence of throws which result in the same outcome, so that, for example, the sequence $HHTHTTH$ contains five runs. Show that the expected number of runs is $1 + 2(n-1)p(1-p)$. Find the variance of the number of runs.

*Solution:* Let $X_i$ be the indicator function of the event that the outcome of the $i+1$th toss is different from the outcome of the $i$th toss. Let $X$ denote the number of distinct runs, which can be expressed in terms of indicators as follows: $X = 1 + \sum_{i=1}^{n-1} X_i$. Thus the expectation is given by:

$$E(X) = 1 + (n-1)E(X_i)$$

$$E(X) = 1 + 2(n-1)p(1-p)$$

**Problem 3 (Urn Removal):** An urn contains $n$ balls numbered $1, \ldots, n$. We remove $k$ balls at random (without replacement) and add up their numbers. Find the mean and variance of the total.

*Solution:* Let $X_i$ denote the number shown on the $i$th ball and note that $E(X_i) = \frac{n+1}{2}$. Then the total for the $k$ removed balls is given by $X = \sum_{i=1}^{k} X_i$, which has expectation given by $E(X) = \sum_{i=1}^{k} E(X_i) = \frac{k(n+1)}{2}$.
We use the following formula to compute the variance: $\mathrm{Var}(X) = E(X^2) - (E(X))^2$. Note that $(E(X))^2 = \frac{k^2(n+1)^2}{4}$. So all that remains is to compute $E(X^2)$:

$$\begin{aligned}
E(X^2) &= E\left[\left(\sum_{i=1}^{k} X_i\right)^2\right] \\
&= E\left[\sum_{i=1}^{k} X_i^2 + \sum_{i \neq j} X_i X_j\right] \\
&= \sum_{i=1}^{k} E(X_i^2) + \sum_{i \neq j} E(X_i X_j) \\
&= kE(X_1^2) + 2\binom{k}{2} E(X_1 X_2) \\
&= k\left(\frac{1}{n}\sum_{j=1}^{n} j^2\right) + k(k-1)E(X_1 X_2) \\
&= \frac{k}{n}\left(\sum_{j=1}^{n} j^2\right) + k(k-1)E(X_1 X_2) \\
&= \frac{k}{n}\left(\sum_{j=1}^{n} j^2\right) + \frac{k(k-1)}{n(n-1)} 2\sum_{i \neq j} ij \\
&= \frac{k}{n}\left(\frac{1}{3}n(n+1)(n+2) - \frac{1}{2}n(n+1)\right) + \frac{k(k-1)}{n(n-1)} 2\sum_{i \neq j} ij \\
&= \frac{k}{n}\left(\frac{1}{3}n(n+1)(n+2) - \frac{1}{2}n(n+1)\right) + \frac{k(k-1)}{n(n-1)} 2\sum_{j=1}^{n} j(n(n+1) - j(j+1)) \\
&= \frac{1}{6}k(n+1)(2n+1) + \frac{1}{12}k(k-1)(3n+2)(n+1)
\end{aligned}$$

From here we have:

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{1}{12}(n+1)k(n-k)$$

**Problem 4 (Simpson's Game):** You have two bags, each with a mixture of green and red marbles. You can choose a bag, then randomly pick a marble from the bag. If it's green you win \$100, but if it's red, you get nothing.

**Game 1:** Bag $A$ contains 5 green marbles and 6 red marbles. Bag $B$ has 3 green marbles and 4 red marbles. What bag should you choose to increase your odds of winning?

**Game 2:** Bag $C$ contains 6 green marbles and 3 red marbles. Bag $D$ has 9 green marbles and 5 red marbles. What bag should you choose to increase your odds of winning?

**Game 3:** We combine bags $A$ and $C$ into bag $E$ and we combine bags $B$ and $D$ into bag $F$. What bag should you choose to increase your odds of winning?

*Solution:*
**Game 1:** Since there is a $\frac{5}{11} > \frac{3}{7}$ chance of winning with bag $A$, you should pick bag $A$.

**Game 2:** Since there is a $\frac{6}{9} > \frac{9}{14}$ chance of winning with bag $C$, you should pick bag $C$.

**Game 3:** Naively, you might think that bag $E$ should have higher odds of winning than bag $F$. However this is wrong notice that the probability of winning with bag $F$ is $\frac{12}{21} > \frac{11}{20}$, which seems counterintuitive, because bags $A$ and $C$ win with higher probability than bags $B$ and $D$ respectively, but when combined bag $E$ wins with lower probability than bag $F$.