Stats100A **Summer 2024**

Week 3: Challenge Problems 3 - Solutions

Author: Andrew Lizarraga

Solutions

Problem 1 (Poker Dice): I have five fair dice, each with the following faces: $9, 10, J, Q, K, A$. When rolling these 5 dices, what's the probability of 2 pairs?

Solution: The total number of possible outcomes is 6^5. Now we need to consider the number of ways in which there are two pairs. We do this in the following way:

- There are $\binom{6}{2} = 15$ ways to choose 2 ranks for the pairs.
- There is $\binom{4}{1} = 4$ ways to choose 1 rank for the fifth die that is different from the 2 pairs.
- We choose 2 dice out of 5 to be the first pair. There is $\binom{5}{2} = 10$ ways to do this.
- We choose 2 dice out of the remaining 3 for the second pair in $\binom{3}{2} = 3$ ways.

Thus the probability of 2 pairs is given by $\frac{15\times4\times10\times3}{6^5} \approx 0.23$

Problem 2 (The Trapped Miner): A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after 2 hours of travel. The second door leads to a tunnel that returns him to the mine after 3 hours of travel. The third door leads to a tunnel that returns him to the mine after 5 hours of travel. Assuming the miner at any time is equally likely to choose any door, what is the expected amount of time until the miner reaches safety?

Solution: Let T denote the variable for the total time it takes for the miner to escape. Let $D \in \{1, 2, 3\}$ denote the door the miner chooses. Then by the law of total expectation we have:

$$E[T] = E[T|D = 1]P(D = 1) + E[T|D = 2]P(D = 2) + E[T|D = 3]P(D = 3)$$
$$E[T] = E[T|D = 1]\frac{1}{3} + E[T|D = 2]\frac{1}{3} + E[T|D = 3]\frac{1}{3} \quad (\star)$$

Now observe that $E[T|D = 1] = 2$, $E[T|D = 2] = 3 + E[T]$, and $E[T|D = 3] = 5 + E[T]$. By substituting these terms into $(\star)$, we have:

$$E[T] = \frac{1}{3}(2 + (3 + E[T]) + (5 + E[T]))$$

$$3E[T] = 10 + 2E[T]$$

$$E[T] = 10$$

So we expect it to take about 10 hours for the miner to reach safety.

Problem 3 (The First Ace): You are given a well-shuffled standard deck of 52 cards. You start at the top of the deck and turn a card over one-at-a-time. What is the expected number of cards that will be turned over before you see the first Ace?

Solution: You can think of the 4 Aces as dividing the other 48 cards of the deck into 5 groups: The cards before the first Ace, the cards between the first and second Ace, and so on. The expected size of each group is $\frac{48}{5} = 9.6$. Thus we need to turn all of the cards over in the first group, plus the the next card for the first Ace. So the expected number of cards we need to turn over is $1 + 9.6 = 10.6$.

Problem 4 (Dot Joining): There are 2024 dots lying in a circle. Suddenly, each dot either forms an edge to a dot on its left or to it right or forms no edge whatsoever (equal chances of each event occurring). About how many isolated dots are there?

Solution: Let X_i be a random variable associated to the i-th dot on the circle. We let $X_i = 1$ if the dot is isolated, otherwise $X_i = 0$. Note, there is a $\frac{1}{3}$ probability it joins an arc to the dot on it's left, a $\frac{1}{3}$ probability it joins an arc to a dot on it's right, and a $\frac{1}{3}$ probability that it remains isolated. Thus, $X_i \sim Ber(p = \frac{1}{3})$. So the expected number of isolated points is given by:

$$E(X_1 + \cdots X_{2024}) = \sum_{i=1}^{2024} \frac{1}{3} = \frac{2024}{3} \approx 674.76$$

Problem 5 (A Dodecahedral Traversal): The image below depicts the planar graph representation of a dodecahedron. Start at any point. Can you traverse this graph (meet every vertex not necessarily every edge) without retracing your steps or intersecting your path and end up at the same point?

Solution: Consider the image below. Starting at the top vertex in the image and traversing the blue path counterclockwise is a solution.